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THE INTEGRATION BY PARTS FOR THE AP-HENSTOCK INTEGRAL

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Abstract. In this paper we introduce the concept of the AP-Henstock integral and prove the integration by parts formula for the AP-Henstock integral.

1. Introduction and Preliminaries

The Henstock integral of real valued functions was first defined by Henstock in 1963 ([2,3]). It is well known([3]) that the Henstock integral is more powerful and simpler than the Lebesgue and Feynman integrals.

In 1994, R. A. Gordon introduced the AP-Henstock integral which is the extension of the Henstock integral and investigated some properties([3,5]).

In this paper we introduce the concept of the AP-Henstock integral and prove the integration by parts formula for the AP-Henstock integral.

Let E be a measurable set and let x be a real number. The density of E at x is defined by

$$d_xE = \lim_{h\to 0+} \frac{\mu(E\cap(x-h,x+h))}{2h},$$

provided the limit exists. The point x is called a point of density of E if $d_xE = 1$. The E^d represents the set of all $x \in E$ such that x is a point of density of E.

A function $f : [a, b] \to R$ is said to be approximately continuous at $c \in [a, b]$ if there exists a measurable set $E \subset [a, b]$ such that $c \in E^d$ and $f\,|_E$ is continuous at c.

A function $F : [a, b] \to R$ is said to be approximately differentiable at $c \in [a, b]$ if there exists a measurable set $E \subset [a, b]$ such that $c \in E^d$

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and

$$\lim_{x \to c, x \in E} \frac{F(x) - F(c)}{x - c}$$

exists. The approximate derivative of F at c is denoted by $F'_{ap}(c)$.

An approximate neighborhood(or ap-nbd) of $x \in [a, b]$ is a measurable set $S_x \subset [a, b]$ containing x as a point of density. For every $x \in E \subset [a, b]$, choose an ap-nbd $S_x \subset [a, b]$ of x. Then we say that $S = \{S_x : x \in E\}$ is a choice on E. A tagged interval $([u, v], x)$ is said to fine to the choice $S = \{S_x\}$ if $u, v \in S_x$. Let $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ be a finite collection of non-overlapping tagged intervals. If $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is fine to a choice S for each i, then we say that P is an S-fine. Let $E \subset [a, b]$. If P is S-fine and each $\xi_i \in E$, then P is called S-fine on E. If P is S-fine and $[a, b] = \cup_{i=1}^n [u_i, v_i]$, then we say that P is an S-fine Henstock partition of $[a, b]$.

2. Properties of the AP-Henstock Integral

DEFINITION 2.1. ([3]) A function $f : [a, b] \to R$ is AP-Henstock integrable if there exists a real number $A \in R$ such that for each $\epsilon > 0$ there is a choice S on $[a, b]$ such that

$$|\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - A| < \epsilon$$

for each S-fine Henstock partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ of $[a, b]$. In this case, A is called the AP-Henstock integral of f on $[a, b]$, and we write $A = (AH) \int_a^b f$.

THEOREM 2.2. *Let f and g be AP-Henstock integrable functions on $[a, b]$. then $\alpha f + \beta g$ is AP-Henstock integrable on $[a, b]$ and $(AP) \int_a^b (\alpha f + \beta g) = \alpha (AP) \int_a^b f + \beta (AP) \int_a^b g$.*

DEFINITION 2.3. Let $F : [a, b] \to R$ be measurable and let $E \subset [a, b]$. Then the function F is AC on E if for each $\epsilon > 0$ there exists a positive number δ such that $\sum_i^n |F(d_i) - F(c_i)| < \epsilon$ for each non-overlapping finite intervals $\{[c_i, d_i]\}_{i=1}^n$ on $[a, b]$ satisfying $c_i, d_i \in E$ and $\sum_{i=1}^n (d_i - c_i)| < \delta$. F is ACG on E if $F\,|_E$ is continuous on E and E can be expressed as a countable union of sets on each of which F is AC.

DEFINITION 2.4. Let $F : [a, b] \to R$ be measurable and let $E \subset [a, b]$ be measurable. Then F is AC_S on E if for each $\epsilon > 0$ there exist a positive number η and a choice S on $[a, b]$ such that $\sum_i^n |F(I_i)| < \epsilon$ for each $S-$ fine partial partition $D = \{(I_i, \xi_i)\}_{i=1}^n$ on $[a,b]$ satisfying $\xi_i \in E$ and $\sum_{i=1}^n |I_i| < \eta$. The function F is ACG_S on E if E can be expressed as a countable union of sets on each of which F is AC_S.

THEOREM 2.5. *([3]) A function $f : [a,b] \to R$ is AP-Henstock integrable on $[a,b]$ if and only if there exists an ACG_S function F on $[a,b]$ such that $F'_{ap} = f$ almost everywhere on $[a,b]$.*

3. The Integration by parts for the AP-Henstock Integral

To prove the integration by parts for the AP-Henstock integral, we need the following theorem.

THEOREM 3.1. *Let F and G be ACG_S on $[a,b]$. If F and G are bounded on $[a,b]$, then FG is ACG_S on $[a,b]$.*

Proof. Since F is ACG_S on $[a,b]$, there exists a sequence of measurable sets $\{A_n\}$ such that $[a,b] = \cup_{i=1}^{\infty} A_n$ and F is AC_S on each A_n. Since G is ACG_S on $[a,b]$, there exists a sequence of measurable sets $\{B_m\}_{m=1}^{\infty}$ such that $[a,b] = \cup_{m=1}^{\infty} B_m$ and G is AC_S on each B_m. We have

$$[a,b] = \cup_{n=1}^{\infty} \cup_{m=1}^{\infty} (A_n \cap B_m).$$

We rewrite the sequence $\{A_n \cap B_m\}_{n,m \geq 1}$ as $\{E_k\}_{k \geq 1}$. Then obiously F and G are AC_S on each E_k . Now let us show that FG is AC_S on each E_k. Let $| F(t) | \leq M$ and $| G(t) | \leq M$ for each $t \in [a,b]$ and fix k. Let $\epsilon > 0$. Since F is AC_S on E_k, there exist a positive η_1 and a choice S_1 on $[a,b]$ such that

$$\sum_{i=1}^{p} | F(I_i) | < \frac{\epsilon}{2M}$$

for each S_1- fine partial partition $\{(I_i, \xi_i)\}_{i=1}^p$ satisfying $\sum_{i=1}^p | I_i | < \eta_1$ and $\xi_i \in E_k$. Since G is AC_S on E_k, there exists a positive $\eta_2 > 0$ and a choice S_2 on $[a,b]$ such that

$$\sum_{i=1}^{p} | G(J_i) | < \frac{\epsilon}{2M}$$

for each S_2- fine partial partition $\{(J_i, \xi_i)\}_{i=1}^p$ satisfying $\sum_{i=1}^p | I_i | < \eta_2$ and $\xi_i \in E_k$.

Let $S = S_1 \cap S_2$ and let $\eta = \min\{\eta_1, \eta_2\}$. Let $D = \{([c_i, d_i], \xi_i\}_{i=1}^m$ be a $S-$fine partial partition that satisfying $\sum_{i=1}^m (d_i - c_i) < \eta$ and $\xi_i \in E_k$.

Then we have

$$\sum_{i=1}^{m} | F(d_i)G(d_i) - F(c_i)G(c_i) |$$

$$\leq \sum_{i=1}^{m} | F(d_i)G(d_i) - F(c_i)G(d_i) | + \sum_{i=1}^{m} | F(c_i)G(d_i) - F(c_i)G(c_i) |$$

$$\leq \sum_{i=1}^{m} | G(d_i) || F(d_i) - F(c_i) | + \sum_{i=1}^{m} | F(c_i) || G(d_i) - G(c_i) |$$

$$\leq M \sum_{i=1}^{m} | F(d_i) - F(c_i) | + M | \sum_{i=1}^{m} | G(d_i) - G(c_i) |$$

$$< M \frac{\epsilon}{2M} + M \frac{\epsilon}{2M} = \epsilon.$$

Hence, FG is AC_S on E_k.

□

THEOREM 3.2. *Let $f : [a, b] \to R$ be AP-Henstock integrable on $[a, b]$ and let $F(x) = (AH) \int_a^x f$ for each $x \in [a, b]$. If F is bounded on $[a, b]$ and G is bounded ACG_S on $[a, b]$, then fG is AP-Henstock integrable on $[a, b]$ and*

$$(AH) \int_a^b fG = F(b)G(b) - (AH) \int_a^b FG'.$$

Proof. Since F and G are ACG_S on $[a, b]$, FG is ACG_S on $[a, b]$ by Theorem 3.1. Hence, $(FG)'_{ap}$ is AP-Henstock integrable on $[a, b]$. Since F is bounded and measurable on $[a, b]$. F is Lebesgue integrable on $[a, b]$ and $(FG)'_{ap}$ is AP-Henstock integrable on $[a, b]$. Since $fG = (FG)'_{ap} - FG'_{ap}$ almost everywhere on $[a, b]$, fG is AP-Henstock integrable on $[a, b]$ and

$$(AH) \int_a^b fG = (AH) \int_a^b (FG)'_{ap} - (AH) \int_a^b FG'_{ap}$$

$$= F(b)G(b) - (AH) \int_a^b FG'_{ap}.$$

□

COROLLARY 3.3. *Let $f : [a,b] \to R$ be AP-Henstock integrable on $[a,b]$ and let $F(x) = (AH)\int_a^x f$ for each $x \in [a,b]$. If F is bounded on $[a,b]$ and G is AC on $[a,b]$, then fG is AP-Henstock integrable on $[a,b]$ and*

$$(AH)\int_a^b fG = F(b)G(b) - (L)\int_a^b F'dG.$$

Proof. By theorem 3.2, the function fG is AP-Henstock integrable on $[a,b]$. Since F is bounded and measurable on $[a,b]$, F is Lebesgue integrable on $[a,b]$. Also, since G is AC on $[a,b]$, G' is Lebesgue integrable on $[a,b]$ and $(L)\int_a^b FG' = (L)\int_a^b FdG$. Hence, we have

$$(AH)\int_a^b fG = F(b)G(b) - (L)\int_a^b FdG.$$

□

THEOREM 3.4. *Let $f : [a,b] \to R$ be AP-Henstock integrable on $[a,b]$ and let $F(x) = (AH)\int_a^x f$ for each $x \in [a,b]$. If F is bounded on $[a,b]$ and G is ACG_S of bounded variation on $[a,b]$, then fG is AP-Henstock integrable on $[a,b]$ and*

$$(AH)\int_a^b fG = F(b)G(b) - (L)\int_a^b FG'.$$

Proof. Since F is bounded ACG_S on $[a,b]$ and G is bounded variation ACG_S on $[a,b]$, FG is ACG_S on $[a,b]$ by Theorem 3.1. Hence $(FG)'_{ap}$ is AP-Henstock integrable on $[a,b]$. Also, since F is bounded measurable on $[a,b]$ and G' is Lebesgue integrable on $[a,b]$, FG' is Lebesgue integrable on $[a,b]$. Since $fG = (FG)'_{ap} - FG'$ almost everywhere on $[a,b]$, fG is AP-Henstock integrable on $[a,b]$ and

$$(AH)\int_a^b fG = F(b)G(b) - (L)\int_a^b FG'.$$

□

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